CERN-PH-TH/2010-305 LMU-ASC 107/10 TUM-HEP-788/10 IPPP/11/02 DCPT/11/04 January 2011

Theory of $B \to K^{(*)} l^+ l^-$ decays at high q^2 : OPE and quark-hadron duality

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Abstract

We develop a systematic framework for exclusive rare B decays of the type $B \to K^{(*)}l^+l^-$ at large dilepton invariant mass q^2 . It is based on an operator product expansion (OPE) for the required matrix elements of the nonleptonic weak Hamiltonian in this kinematic regime. Our treatment differs from previous work by a simplified operator basis, the explicit calculation of matrix elements of subleading operators, and by a quantitative estimate of duality violation. The latter point is discussed in detail, including the connection with the existence of an OPE and an illustration within a simple toy model.

PACS: 12.15.Mm; 12.39.St; 13.20.He

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1 Introduction

The rare decays $B \to K^{(*)} l^+ l^-$ are among the most important probes of flavour physics. They are potentially sensitive to dynamics beyond the Standard Model (SM) and have been intensely studied in the literature [1]. Measurements have been performed at the B-meson factories [2–6] and at the Fermilab Tevatron [7]. Excellent future prospects for detailed measurements are provided by the LHC experiments ATLAS, CMS, and LHCb at CERN [1], and, in the longer run, by Super Flavour Factories based on e^+e^- colliders [8–11].

The calculability of $B \to K^{(*)}l^+l^-$ decay rates and distributions benefits from the fact that these processes are, to first approximation, semileptonic modes. Correspondingly, the hadronic physics is described by $B \to K^{(*)}$ form factors, which multiply a perturbatively calculable amplitude. This simple picture is not exact because also the nonleptonic weak Hamiltonian at scale $\sim m_b$ has $B \to K^{(*)}l^+l^-$ matrix elements. The prominent example is given by hadronic interactions of the form $(\bar{s}b)(\bar{c}c)$, where the charm quarks annihilate into l^+l^- through a virtual photon. Such charm-loop contributions are more complicated theoretically than the form-factor terms. Even though the charm loops are subdominant numerically in the kinematical regions of interest, they cannot be completely neglected. In particular, the related uncertainty needs to be properly estimated in order to obtain accurate predictions.

We need to distinguish three regions in the dilepton invariant mass q^2 , for which the properties of charm loops are markedly different. For $7\,\mathrm{GeV}^2\lesssim q^2\lesssim 15\,\mathrm{GeV}^2$ the presence of very narrow $c\bar{c}$ resonances leads to huge violations of quark hadron duality [12] and the hadronic backgrounds from $B\to K^{(*)}\psi$, followed by $\psi\to l^+l^-$, dominate the short distance rate by two orders of magnitude. This region in q^2 can be removed by experimental cuts.

For $q^2 \lesssim 7 \,\mathrm{GeV^2}$ the kaon is very energetic and the charm loops can be computed systematically in the heavy-quark limit using QCD factorization for B decays into light-like mesons [13,14]. This approach was first employed for $B \to K^{(*)} l^+ l^-$ in [15]. The results have many applications. A summary with detailed references can be found in [1] (see also [16] for a recent analysis).

The high- q^2 region, $q^2 \gtrsim 15 \,\mathrm{GeV}^2$ has received comparatively little attention. In this case the kaon energy is around a GeV or below, and (soft-collinear) QCD factorization is less justified, becoming invalid close to the endpoint of the spectrum at $q^2 = (m_B - m_K)^2$. On the other hand, the large value of q^2 defines a hard scale for the hadronic contribution to $B \to K^{(*)}l^+l^-$. Consequently an operator product expansion (OPE) can be constructed, which generates an expansion of the amplitude in powers of $E_K/\sqrt{q^2}$ (or $\Lambda_{QCD}/\sqrt{q^2}$). Charm loops, and other hadronic contributions, are thus approximated as effective interactions that are local on the soft scales set by E_K and Λ_{QCD} . This simplifies the computation substantially. In fact, to leading order in the OPE the hadronic contribution reduces to a standard form-factor term. This picture has been first discussed at lowest order in the OPE in [17], where it was applied to the endpoint region of $B \to K^{(*)}l^+l^-$ and $B \to K\pi l^+l^-$. In [18] the OPE was considered in some detail,

including a discussion of power corrections.

In the present paper we formulate the OPE for the high- q^2 region of $B \to K^{(*)} l^+ l^$ from the outset. Although our approach is similar in spirit to the analysis of [18], the concrete implementation is different. We will also go beyond the estimates presented in [18] in several ways. An important difference is that [18] combines the OPE with heavy-quark effective theory (HQET), whereas we prefer to work with b-quark fields in full QCD. The latter formulation has the advantage of a simplified operator basis, which makes the structure of power corrections and their evaluation considerably more transparent. We also retain the kinematical dependence on q^2 in the coefficient functions, rather than expanding it around $q^2 = m_b^2$. We further discuss the issue of quark-hadron duality, which appears relevant because of the existence of $c\bar{c}$ resonance structure in the q^2 region of interest. Violations of duality are effects beyond any finite order in the OPE. Using a resonance model based on a proposal by Shifman, we quantify for the first time the size of duality violations in the high- q^2 region of $B \to K^{(*)}l^+l^-$. Further aspects and new results of our analysis will be summarized in sections 8 and 9. The main conclusion is that $B \to K^{(*)}l^+l^-$ is under very good theoretical control also for $q^2 \gtrsim 15 \,\mathrm{GeV}^2$. Precise predictions can be obtained in terms of the standard form factors, with essentially negligible effects from the additional hadronic parameters related to power corrections and duality violation.

The paper is organized as follows. Section 2 collects basic expressions for later reference. In section 3 our formulation of the OPE for $B \to K^{(*)}l^+l^-$ at high q^2 is described and the power expansion is constructed explicitly, complete to second order in $1/\sqrt{q^2}$ and with a discussion of weak annihilation as an example of a (small) third-order correction. In section 4 we present an estimate of the matrix elements of higher-dimensional operators and quantify their impact on the decay amplitudes for both $B \to K$ and $B \to K^*$ transitions. Section 5 discusses the connection between the OPE for large q^2 and QCD factorization for energetic kaons, which are shown to give consistent results at intermediate $q^2 \approx 15 \, \text{GeV}^2$. In section 6 we address the subject of duality violation in the context of a toy model analysis. The estimate is then adapted to the case of $B \to K l^+ l^-$ in section 7. In this section we also address conceptual aspects relevant for the existence of the OPE and the notion of quark-hadron duality. A comparison of our approach with the literature is given in section 8 before we conclude in section 9. Details on the basis of operators in the OPE are described in appendix A and some numerical input is collected in appendix B.

2 Basic formulas

2.1 Weak Hamiltonian

The effective Hamiltonian for $b \to sl^+l^-$ transitions reads [19,20,21]

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p \left[C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3,\dots,10} C_i Q_i \right]$$
 (1)

where

$$\lambda_p = V_{ps}^* V_{pb} \tag{2}$$

The operators are given by

$$Q_{1}^{p} = (\bar{p}b)_{V-A}(\bar{s}p)_{V-A}, \qquad Q_{2}^{p} = (\bar{p}_{i}b_{j})_{V-A}(\bar{s}_{j}p_{i})_{V-A},$$

$$Q_{3} = (\bar{s}b)_{V-A} \sum_{q} (\bar{q}q)_{V-A}, \qquad Q_{4} = (\bar{s}_{i}b_{j})_{V-A} \sum_{q} (\bar{q}_{j}q_{i})_{V-A},$$

$$Q_{5} = (\bar{s}b)_{V-A} \sum_{q} (\bar{q}q)_{V+A}, \qquad Q_{6} = (\bar{s}_{i}b_{j})_{V-A} \sum_{q} (\bar{q}_{j}q_{i})_{V+A},$$

$$Q_{7} = \frac{e}{8\pi^{2}} m_{b} \bar{s}\sigma_{\mu\nu} (1 + \gamma_{5}) F^{\mu\nu}b, \qquad Q_{8} = \frac{g}{8\pi^{2}} m_{b} \bar{s}\sigma_{\mu\nu} (1 + \gamma_{5}) G^{\mu\nu}b,$$

$$Q_{9} = \frac{\alpha}{2\pi} (\bar{s}b)_{V-A} (\bar{l}l)_{V}, \qquad Q_{10} = \frac{\alpha}{2\pi} (\bar{s}b)_{V-A} (\bar{l}l)_{A}$$

$$(3)$$

Note that the numbering of $Q_{1,2}^p$ is reversed with respect to the convention of [19]. Our coefficients $C_{9,10}$ correspond to $\tilde{C}_{9,10}$ in [19] and we include the factor of $\alpha/(2\pi)$ in the definition of $Q_{9,10}$. The sign conventions for the electromagnetic and strong couplings correspond to the covariant derivative $D_{\mu} = \partial_{\mu} + ieQ_{f}A_{\mu} + igT^{a}A_{\mu}^{a}$. With these definitions the coefficients $C_{7,8}$ are negative in the Standard Model.

2.2 Dilepton-mass spectra and short-distance coefficients

We define the kinematic quantities $s=q^2/m_B^2$ (where q^2 is the dilepton invariant mass squared), $r_K=m_K^2/m_B^2$, and

$$\lambda_K(s) = 1 + r_K^2 + s^2 - 2r_K - 2s - 2r_K s \tag{4}$$

The differential branching fractions for $\bar{B} \to \bar{K} l^+ l^-$ can then be written as [22]

$$\frac{dB(\bar{B} \to \bar{K}l^+l^-)}{ds} = \tau_B \frac{G_F^2 \alpha^2 m_B^5}{1536\pi^5} |V_{ts}V_{tb}|^2 \cdot \lambda_K^{3/2}(s) f_+^2(s) \left(|a_9(Kll)|^2 + |a_{10}(Kll)|^2 \right)$$
(5)

The coefficient $a_9(Kll)$ contains the Wilson coefficient $C_9(\mu)$ combined with the shortdistance parts of the $\bar{B} \to \bar{K}l^+l^-$ matrix elements of operators Q_1, \ldots, Q_8 . The coefficient $a_9(Kll)$ multiplies the matrix element of the local operator Q_9 in the decay amplitude. The coefficient $a_{10}(Kll) = C_{10}$ of the operator Q_{10} is determined by very short distances $\sim 1/M_W$ and is precisely known.

The corresponding formulas for $\bar{B} \to \bar{K}^* l^+ l^-$ can for instance be found in [23].

3 OPE for $B \to M l^+ l^-$ amplitudes at high q^2

3.1 General structure

The amplitudes for the exclusive decays $B \to M l^+ l^-$, where M = K, K^* , or a similar meson, are given by the matrix element of the effective Hamiltonian in (1) between the

initial B meson and the Ml^+l^- final state. The dominant contribution comes from the semileptonic operators $Q_{9,10}$. Their matrix elements are simple in the sense that all hadronic physics is described by a set of $B \to M$ transition form factors. This is also true for the electromagnetic operator Q_7 . The matrix elements of the hadronic operators Q_1, \ldots, Q_6, Q_8 are more complicated. They are induced by photon exchange and can be expressed through the matrix element of a correlator between the hadronic part of the effective Hamiltonian

$$H^p \equiv C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3}^{6,8} C_i Q_i$$
 (6)

and the electromagnetic current of the quarks

$$j^{\mu} \equiv Q_q \, \bar{q} \gamma^{\mu} q \tag{7}$$

where Q_q is the electric charge quantum number of quark flavour q and a summation over q is understood. The decay amplitude may thus be written as

$$A(\bar{B} \to \bar{M}l^+l^-) = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi} \lambda_t \left[\left(A_9^{\mu} + \frac{\lambda_u}{\lambda_t} A_{cu}^{\mu} \right) \bar{u} \gamma_{\mu} v + A_{10}^{\mu} \bar{u} \gamma_{\mu} \gamma_5 v \right]$$
(8)

where \bar{u} and v are the lepton spinors and

$$A_{9}^{\mu} = C_{9} \langle \bar{M} | \bar{s} \gamma^{\mu} (1 - \gamma_{5}) b | \bar{B} \rangle - \frac{8\pi^{2}}{q^{2}} i \int d^{4}x \, e^{iq \cdot x} \, \langle \bar{M} | T \, j^{\mu}(x) H^{c}(0) | \bar{B} \rangle$$

$$+ C_{7} \frac{2im_{b}}{q^{2}} q_{\lambda} \, \langle \bar{M} | \bar{s} \sigma^{\lambda \mu} (1 + \gamma_{5}) b | \bar{B} \rangle$$

$$A_{cu}^{\mu} = \frac{8\pi^{2}}{q^{2}} i \int d^{4}x \, e^{iq \cdot x} \, \langle \bar{M} | T \, j^{\mu}(x) (H^{u}(0) - H^{c}(0)) | \bar{B} \rangle$$

$$A_{10}^{\mu} = C_{10} \, \langle \bar{M} | \bar{s} \gamma^{\mu} (1 - \gamma_{5}) b | \bar{B} \rangle$$
(9)

For $b \to s$ transitions the contribution from A_{cu}^{μ} is suppressed by the prefactor λ_u/λ_t and can be neglected.

Exploiting the presence of the large scale $q^2 \sim m_b^2$, an operator product expansion (OPE) can be performed for the non-local term

$$\mathcal{K}_{H}^{\mu}(q) \equiv -\frac{8\pi^{2}}{q^{2}} i \int d^{4}x \, e^{iq \cdot x} \, T \, j^{\mu}(x) H^{c}(0)$$
 (10)

which describes the contribution of 4-quark operators to the $b \to sl^+l^-$ amplitude. Such an OPE corresponds to integrating out the hard quark loop, leading to a series of local effective interactions for the high- q^2 region. To leading order in the large- q^2 expansion this has been presented in [17]. A discussion of the OPE including higher-order contributions has been given in [18].

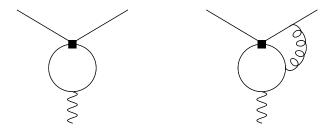


Figure 1: OPE for the correlation function \mathcal{K}_{H}^{μ} : Leading-power contributions (operators of dimension 3). The solid square and the virtual-photon attachment indicate the insertion of the weak Hamiltonian and of the electromagnetic current, respectively. The diagram on the left shows the lowest-order term in QCD. On the right is a sample diagram for the next-to-leading QCD corrections of order α_s .

Before going into more detail we discuss the basic structure of the OPE for \mathcal{K}_H^{μ} . The expansion may be written as

$$\mathcal{K}_{H}^{\mu}(q) = \sum_{d,n} C_{d,n}(q) \, \mathcal{O}_{d,n}^{\mu} \tag{11}$$

The operators $\mathcal{O}_{d,n}$ are composed of quark and gluon fields and have the flavour quantum numbers of $(\bar{s}b)$. They are ordered according to their dimension d and carry an index n labeling different operators with the same dimension. The $C_{d,n}(q)$ are the corresponding Wilson coefficients, which can be computed in perturbation theory. The large scales justifying the expansion are m_b^2 and q^2 . They are counted as quantities of the same order. The coefficients then scale as $C_{d,n} \sim m_b^{3-d}$ in the heavy-quark limit. Since the matrix elements $\langle \bar{M}|\mathcal{O}_{d,n}|\bar{B}\rangle$ scale as $\sqrt{m_b}$, the matrix element of each term in (11) behaves as $m_b^{7/2-d}$. Current conservation implies that all operators are transverse in q,

$$q_{\mu}\mathcal{O}_{d,n}^{\mu} \equiv 0 \tag{12}$$

It is convenient to work with the b-quark field in full QCD. This field could be further expanded within heavy-quark effective theory (HQET), in order to make the m_b -dependence fully explicit. In such an approach many additional operators would arise whose hadronic matrix elements are not readily known. In contrast, the advantage of using the b-field in full QCD is that fewer operators appear and that the matrix elements of the leading ones are given by common form factors. In this method the OPE becomes particularly transparent and we will adopt it here.

At leading order in the OPE (d=3), illustrated in Fig. 1, and in the chiral limit $(m_s=0)$ there are two operators

$$\mathcal{O}_{3,1}^{\mu} = \left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2}\right) \bar{s}\gamma_{\nu}(1 - \gamma_5)b \tag{13}$$

$$\mathcal{O}_{3,2}^{\mu} = \frac{im_b}{q^2} q_\lambda \,\bar{s}\sigma^{\lambda\mu} (1 + \gamma_5) b \tag{14}$$

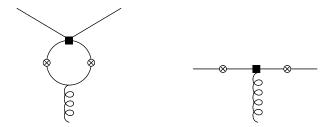


Figure 2: OPE for the correlation function \mathcal{K}_{H}^{μ} : Second-order power corrections (operators of dimension 5). The crossed circles denote the places where the virtual photon can be attached.

Using the equations of motion for the external quarks it can be shown that all possible bilinears $\bar{s}_L\Gamma b$ and $\bar{s}D\Gamma b$ arising from the correlator \mathcal{K}_H^{μ} can be expressed in terms of (13) and (14). Consequently, no independent dimension-4 operators of the form $\bar{s}D\Gamma b$ can appear in the OPE. The complete proof is given in appendix A. As an example, the operator $\bar{s}iD^{\mu}(1+\gamma_5)b$ satisfies the equations-of-motion identity (for $m_s=0$)

$$\bar{s}i\overleftarrow{D}^{\mu}(1+\gamma_5)b \equiv -\frac{m_b}{2}\bar{s}\gamma^{\mu}(1-\gamma_5)b + \frac{1}{2}\partial_{\nu}(\bar{s}\sigma^{\mu\nu}(1+\gamma_5)b) + \frac{i}{2}\partial^{\mu}(\bar{s}(1+\gamma_5)b)$$
 (15)

For any $\bar{B} \to X_s$ matrix element with momentum transfer q this is equivalent to

$$\bar{s}i\overleftarrow{D}^{\mu}(1+\gamma_5)b = -\frac{m_b}{2}\bar{s}\gamma^{\mu}(1-\gamma_5)b - \frac{i}{2}q_{\nu}\bar{s}\sigma^{\mu\nu}(1+\gamma_5)b + \frac{1}{2}q^{\mu}\bar{s}(1+\gamma_5)b \qquad (16)$$

Because of current conservation only the transverse part $(g_{\mu\lambda} - q_{\mu}q_{\lambda}/q^2)O^{\lambda}$ of such an operator O^{μ} can appear in the OPE. From (16) we see that this part can be reduced to a linear combination of (13) and (14).

If we keep $m_s \neq 0$, two additional operators have to be considered

$$\mathcal{O}_{4,1}^{\mu} = m_s \left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2} \right) \bar{s}\gamma_{\nu} (1 + \gamma_5) b \tag{17}$$

$$\mathcal{O}_{4,2}^{\mu} = \frac{im_s m_b}{g^2} q_{\lambda} \,\bar{s} \sigma^{\lambda \mu} (1 - \gamma_5) b \tag{18}$$

Since m_s/m_b is small, and numerically similar to Λ/m_b , we may formally count these as operators of dimension 4. Because they are absent at order α_s^0 , their impact will be suppressed to the level of $\alpha_s m_s/m_b \sim 0.5\%$, which is negligible. Note that these operators do in any case not introduce new hadronic form factors.

At d=5 (Fig. 2) we encounter operators with a factor of the gluon field strength $G_{\mu\nu}$, which have the form

$$\mathcal{O}_{5,n}^{\mu} = \bar{s}(gG\Gamma_n)^{\mu}b\tag{19}$$

where the Γ_n denote Dirac and Lorentz structures. We will treat the OPE explicitly to the level of d=5, that is including power corrections up to second order in Λ/m_b .

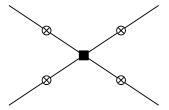


Figure 3: OPE for the correlation function \mathcal{K}_{H}^{μ} : Weak annihilation as an example of third-order power corrections (operators of dimension 6). Crossed circles denote the various virtual photon attachments.

Although we will not give a full treatment of dimension-6 corrections, we consider as an example the effect of weak annihilation (Fig. 3). This contribution is characterized by the annihilation of the two valence quarks in the \bar{B} meson in the $\bar{B} \to \bar{M}$ transition through the weak Hamiltonian. It is described by 4-quark operators, which read schematically

$$\mathcal{O}^{\mu}_{6ann\ n} = (\bar{r}\Gamma_1 b\,\bar{s}\Gamma_2 r)^{\mu}_n \tag{20}$$

with Lorentz and Dirac structures indicated by $(\Gamma_{1,2})_n$, and the light quark field r = u, d in the case of non-strange \bar{B} mesons. Weak annihilation provides a mechanism to break isospin symmetry, directly at the level of the transition operator. In \mathcal{K}_H^{μ} weak annihilation, in addition to being a third order power correction, comes only from QCD penguin operators, which have small coefficients. The contribution to isospin breaking from this source will therefore be strongly suppressed.

3.2 OPE to leading order in α_s

In this section we give explicitly the first few terms in the OPE to leading order in renormalization-group improved perturbation theory, that is neglecting relative corrections of $\mathcal{O}(\alpha_s)$. This order for \mathcal{K}_H is relevant in the next-to-leading logarithmic approximation to the $\bar{B} \to \bar{M} l^+ l^-$ amplitude. We may then write

$$\mathcal{K}_{H}^{\mu} = \mathcal{K}_{H3}^{\mu} + \mathcal{K}_{H5}^{\mu} + \mathcal{K}_{H6a}^{\mu} + \mathcal{O}(\alpha_{s}, (\Lambda/m_{b})^{3})$$
(21)

The lower indices of the terms on the r.h.s. denote the dimension d of the corresponding local operators, which come with a coefficient of order $1/m_b^{d-3}$. The first term reads

$$\mathcal{K}_{H3}^{\mu} = \left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2}\right) \bar{s}\gamma_{\nu}(1 - \gamma_5)b \cdot \left[h(z,\hat{s})(C_1 + 3C_2 + 3C_3 + C_4 + 3C_5 + C_6) - \frac{1}{2}h(1,\hat{s})(4C_3 + 4C_4 + 3C_5 + C_6) - \frac{1}{2}h(0,\hat{s})(C_3 + 3C_4) + \frac{2}{9}(3C_3 + C_4 + 3C_5 + C_6)\right]$$
(22)

The coefficient in (22) requires a UV renormalization, which has to be consistent with the definition of C_9 . The expression given here corresponds to the NDR scheme used in [19]. The function $h(z, \hat{s})$ is $(z \equiv m_c/m_b, \hat{s} \equiv q^2/m_b^2, x \equiv 4z^2/\hat{s} = 4m_c^2/q^2)$

$$h(z,\hat{s}) = -\frac{8}{9} \ln \frac{m_b}{\mu} - \frac{8}{9} \ln z + \frac{8}{27} + \frac{4}{9} x + \frac{2}{9} (2+x) \sqrt{1-x} \left(\ln \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} + i\pi \right)$$
(23)

Next we have

$$\mathcal{K}_{H5}^{\mu} = \left[\varepsilon^{\alpha\beta\lambda\rho} \frac{q_{\beta}q^{\mu}}{q^{2}} + \varepsilon^{\beta\mu\lambda\rho} \frac{q_{\beta}q^{\alpha}}{q^{2}} - \varepsilon^{\alpha\mu\lambda\rho} \right] \bar{s}\gamma_{\lambda}(1 - \gamma_{5}) gG_{\alpha\rho} b \frac{C_{1}Q_{c}}{q^{2}} f(x)
- \frac{q_{\lambda}}{m_{B}} \bar{s}gG_{\alpha\beta} (g^{\alpha\lambda}\sigma^{\beta\mu} - g^{\alpha\mu}\sigma^{\beta\lambda}) (1 + \gamma_{5}) b \frac{4C_{8}Q_{b}}{q^{2}}$$
(24)

Here

$$f(x) = \frac{x}{\sqrt{1-x}} \left(\ln \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} + i\pi \right) - 2$$
 (25)

with $x = 4m_c^2/q^2$. The charm-loop contribution in (24), proportional to C_1 , can be inferred from [24]. Note that here we use the convention $\varepsilon^{0123} = -1$. In writing (24) we have neglected terms with the small QCD penguin coefficients C_3, \ldots, C_6 .

Finally, weak-annihilation diagrams give the dimension-6 term

$$\mathcal{K}_{H6a}^{\mu} = \frac{8\pi^{2}}{q^{4}} q_{\lambda} \sum_{r=u,d} \left[2Q_{r}(\bar{r}_{i}\gamma^{\mu}(1-\gamma_{5})b_{j}\,\bar{s}_{k}\gamma^{\lambda}(1-\gamma_{5})r_{l} - \{\mu \leftrightarrow \lambda\}) \right. \\
\left. - \frac{2}{3}i\varepsilon^{\mu\lambda\beta\nu}\bar{r}_{i}\gamma_{\beta}(1-\gamma_{5})b_{j}\,\bar{s}_{k}\gamma_{\nu}(1-\gamma_{5})r_{l} \right] \left(\delta_{ij}\delta_{kl}C_{4} + \delta_{il}\delta_{kj}C_{3}\right) \\
+ \frac{16\pi^{2}i}{q^{4}}q_{\lambda} \sum_{r=u,d} \left[Q_{r}(\bar{r}_{i}(1-\gamma_{5})b_{j}\,\bar{s}_{k}\sigma^{\mu\lambda}(1+\gamma_{5})r_{l} + \bar{r}_{i}\sigma^{\mu\lambda}(1-\gamma_{5})b_{j}\,\bar{s}_{k}(1+\gamma_{5})r_{l}) \right. \\
\left. - \frac{1}{3}(\bar{r}_{i}(1-\gamma_{5})b_{j}\,\bar{s}_{k}\sigma^{\mu\lambda}(1+\gamma_{5})r_{l} - \bar{r}_{i}\sigma^{\mu\lambda}(1-\gamma_{5})b_{j}\,\bar{s}_{k}(1+\gamma_{5})r_{l}) \right] \\
\left. \left. \left(\delta_{ij}\delta_{kl}C_{6} + \delta_{il}\delta_{kj}C_{5}\right) \right. \right. \tag{26}$$

The terms in (26) only arise from QCD penguin operators, which have small coefficients. We remark that all operators in (22), (24) and (26) vanish identically when contracted with q_{μ} , as required by gauge invariance.

3.3 $\mathcal{O}(\alpha_s)$ corrections to the charm loop

The non-factorizable $\mathcal{O}(\alpha_s)$ corrections to the charm loop arise from diagrams like the one shown on the r.h.s. of Fig. 1. The q^2 -dependence has been recently calculated in analytic form as a Taylor expansion in the small parameter $z = m_c^2/m_b^2$ [25]. Analytic

results for $m_c = 0$ had been presented in [26]. In the kinematical range relevant to our considerations, it has been shown that the convergence of the series is very good. We therefore use the MATHEMATICA input files provided by the authors of [25] in the online preprint publication for a numerical estimate. We find that the non-factorizable $\mathcal{O}(\alpha_s)$ corrections to the charm loop lead to a 10-15% reduction of the real part of a_9 and contribute a negative imaginary part of again 10-15% relative to the short-distance contribution from C_9 (the precise value is scheme-dependent). This is in agreement with the effect found for the inclusive $B \to X_s l^+ l^-$ rate in the high- q^2 region, as discussed in [25], and is similar to the effect observed for the low- q^2 region in the exclusive decay modes, see Table 5 in [27].

It is to be stressed that these corrections almost compensate the factorizable charm-loop contribution (diagram on the l.h.s. in Fig. 1). The reason why the $\mathcal{O}(\alpha_s)$ corrections are not suppressed stems from the different colour structure of the diagrams. Whereas the factorizable charm loop comes with a colour-suppressed combination of Wilson coefficients, the additional gluon exchange allows the $c\bar{c}$ -pair to be in a colour-octet state with no such suppression. At even higher orders in perturbation theory, $\mathcal{O}(\alpha_s^n)$ with $n \geq 2$, on the other hand, the numerical effect on a_9 should really be small, as no new additionally enhanced colour structures will arise.

4 Matrix elements and power corrections

The computation of the amplitude from the OPE requires the evaluation of the matrix elements of the local operators. We estimate in particular the matrix elements of the higher-dimensional contributions. This will allow us to quantify power corrections to the $B \to K^{(*)} l^+ l^-$ amplitude at high q^2 . The cases of $B \to K$ and $B \to K^*$ transitions will be considered in turn.

$4.1 \quad B \rightarrow K$

The matrix element of the leading dimension-3 operator is given in terms of the familiar form factors f_{\pm} , defined by (p = k + q)

$$\langle \bar{K}(k)|\bar{s}\gamma^{\mu}(1-\gamma_{5})b|\bar{B}(p)\rangle = 2f_{+}(q^{2})k^{\mu} + [f_{+}(q^{2}) + f_{-}(q^{2})]q^{\mu}$$
(27)

At the level of the dimension-5 correction in (24) one encounters operators of the form $\bar{s}G_{\alpha\rho}\Gamma b$. Their matrix elements introduce, in general, new nonperturbative form factors. Using Lorentz invariance and the antisymmetry of $G_{\alpha\beta}$ and $\sigma^{\rho\tau}$ one can show that

$$q_{\lambda} \langle \bar{K}(k) | \bar{s} G_{\alpha\beta} (g^{\alpha\lambda} \sigma^{\beta\mu} - g^{\alpha\mu} \sigma^{\beta\lambda}) (1 + \gamma_5) b | \bar{B}(p) \rangle \equiv 0$$
 (28)

In order to estimate the remaining term we assume $\Lambda \ll E_K \ll m_B$ for the kaon energy E_K . In this limit the matrix element can be computed in QCD factorization. To leading order we then find

$$\langle \bar{K}(k)|\mathcal{K}_{H5}^{\mu}|\bar{B}(p)\rangle = -\frac{\pi\alpha_s(E_K)C_F}{N}C_1Q_cf(x)\frac{m_Bf_Bf_K}{\lambda_Bq^2}\left[k^{\mu} - \frac{k\cdot q}{q^2}q^{\mu}\right]$$
(29)

where $C_F = (N^2 - 1)/(2N)$, N is the number of colours, and $1/\lambda_B$ is the first inverse moment of the B-meson light-cone distribution amplitude. This matrix element scales as $(\Lambda/m_b)^2$ relative to (27) in the heavy quark limit, and as $\mathcal{O}(1)$ for large N.

In a similar way we can estimate the weak annihilation term

$$\langle \bar{K}(k)|\mathcal{K}^{\mu}_{H6a}|\bar{B}(p)\rangle = -\frac{16\pi^2 Q_r f_B f_K}{q^2} \left(C_4 + \frac{C_3}{3}\right) \left[k^{\mu} - \frac{k \cdot q}{q^2} q^{\mu}\right]$$
 (30)

This contribution is power-suppressed as $(\Lambda/m_b)^3$ relative to (27). The suppression by the small Wilson coefficients $C_{3,4}$ is partly compensated by a large numerical factor of π^2 . To relative order $(\Lambda/m_b)^3$ there is no contribution from the term with C_5 and C_6 . Note that the result in (30) also corresponds to the matrix element obtained when naively factorizing the four-quark operators.

Normalized to the amplitude coefficient $a_9 = C_9 + \ldots$, the power corrections from \mathcal{K}_{H5} and \mathcal{K}_{H6a} read

$$\Delta a_{9,H5}(K) = -\frac{\pi \alpha_s(E_K)C_F}{2N} C_1 Q_c f(x) \frac{m_B f_B f_K}{\lambda_B f_+(q^2) q^2}$$
(31)

$$\Delta a_{9,H6a}(K) = -\left(C_4 + \frac{C_3}{3}\right) \frac{8\pi^2 Q_r f_B f_K}{f_+(q^2) q^2}$$
(32)

where r = u, d refers to the spectator quark in the B meson.

Numerically, we find $\Delta a_{9,H5}(K) = 0.019 - 0.012i$ at $q^2 = 15 \,\text{GeV}^2$ for central values of the parameters. This number comes with a substantial uncertainty, in particular from λ_B and f_+ . Nevertheless, the correction to $a_9 \approx 4$ is very small, most likely below 1% in magnitude. The correction in (31) diminishes further for larger q^2 , reaching $\Delta a_{9,H5}(K) = 0.006 - 0.002i$ at the endpoint. Towards the endpoint the kaon becomes soft and the result in (31), based on $E_K \gg \Lambda_{QCD}$, can only be viewed as a rough model calculation. The conclusion that $\Delta a_{9,H5}(K)$ remains negligibly small should however still hold. The weak annihilation correction is $\Delta a_{9,H6a}(K) = 0.003$ at $q^2 = 15 \,\text{GeV}^2$ for r = u and therefore entirely negligible, a consequence also of the small Wilson coefficients.

$4.2 \quad B \rightarrow K^*$

In the case of the decay into a vector meson the relevant form factors are defined as $(m_V = m_{K^*})$

$$\left\langle \bar{K}^*(k,\varepsilon) \middle| \bar{s}\gamma^{\mu} b \middle| \bar{B}(p) \right\rangle = -2i \frac{V(q^2)}{m_B + m_V} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\nu}^* p_{\rho} k_{\sigma}$$
(33)

$$\langle \bar{K}^*(k,\varepsilon) | \bar{s}\gamma^{\mu}\gamma_5 b | \bar{B}(p) \rangle = 2m_V A_0(q^2) \frac{\varepsilon^* \cdot q}{q^2} q^{\mu} + (m_B + m_V) A_1(q^2) \left[\varepsilon^{*\mu} - \frac{\varepsilon^* \cdot q}{q^2} q^{\mu} \right]$$

$$- A_2(q^2) \frac{\varepsilon^* \cdot q}{m_B + m_V} \left[(p+k)^{\mu} - \frac{m_B^2 - m_V^2}{q^2} q^{\mu} \right]$$
(34)

It is convenient to treat the decay into longitudinally and transversely polarized vector mesons separately. Omitting terms proportional to q^{μ}

$$\langle \bar{K}_{\parallel}^{*}(k,\varepsilon)|\bar{s}\gamma^{\mu}(1-\gamma_{5})b|\bar{B}(p)\rangle =$$

$$-2k^{\mu} \left[\frac{m_B + m_V}{2m_V} A_1 \frac{1 - m_V^2 / (m_B E)}{\sqrt{1 - (m_V / E)^2}} - \frac{m_B E \sqrt{1 - (m_V / E)^2}}{m_V (m_B + m_V)} A_2 \right]$$
(35)

$$\langle \bar{K}_{\perp}^{*}(k,\varepsilon)|\bar{s}\gamma^{\mu}(1-\gamma_{5})b|\bar{B}(p)\rangle = \frac{-2iV}{m_{B}+m_{V}}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\perp\nu}^{*}p_{\rho}k_{\sigma} - (m_{B}+m_{V})A_{1}\varepsilon_{\perp}^{*\mu}$$
(36)

In the large energy limit $E \gg m_V$, which we may use in the normalization of the power corrections, (35) and (36) simplify to [28,29]

$$\langle \bar{K}_{\parallel}^{*}(k,\varepsilon)|\bar{s}\gamma^{\mu}(1-\gamma_{5})b|\bar{B}(p)\rangle = -2k^{\mu}A_{0}$$

$$\langle \bar{K}_{\perp}^{*}(k,\varepsilon)|\bar{s}\gamma^{\mu}(1-\gamma_{5})b|\bar{B}(p)\rangle = -\frac{2V}{m_{B}}\left(i\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\perp\nu}^{*}p_{\rho}k_{\sigma} + k\cdot p\,\varepsilon_{\perp}^{*\mu}\right)$$
(37)

The case of a longitudinally polarized K^* is very similar to the case of a pseudoscalar K, discussed in section 4.1, and we find

$$\Delta a_{9,H5}(K_{\parallel}^*) = -\frac{\pi \alpha_s(E_K)C_F}{2N} C_1 Q_c f(x) \frac{m_B f_B f_{\parallel}}{\lambda_B A_0(q^2) q^2}$$
(38)

where f_{\parallel} is the decay constant of K_{\parallel}^* .

For a K^* with transverse polarization we obtain

$$\Delta a_{9,H5}(K_{\perp}^*) = -\frac{\pi \alpha_s(E_K)C_F}{4N} \frac{m_B f_B f_{\perp}}{\lambda_B V(q^2) q^2} \left(C_1 Q_c f(x) + 8C_8 Q_b \right)$$
(39)

where f_{\perp} is the decay constant of K_{\perp}^* .

Numerically we have $\Delta a_{9,H5}(K_{\parallel}^*) = 0.021 - 0.012i$ and $\Delta a_{9,H5}(K_{\perp}^*) = 0.008 - 0.006i$ at $q^2 = 15 \,\text{GeV}^2$ for our standard set of parameters. These corrections are of similar size as for the pseudoscalar kaon and they are likewise negligible.

Finally, we quote the corrections from weak annihilation

$$\Delta a_{9,H6a}(K_{\parallel}^{*}) = -\left(C_4 + \frac{C_3}{3}\right) \frac{8\pi^2 Q_r f_B f_{\parallel}}{A_0(q^2)q^2} \tag{40}$$

$$\Delta a_{9,H6a}(K_{\perp}^{*}) = -\left(C_6 + \frac{C_5}{3}\right) \frac{8\pi^2 f_B f_{\perp} m_B^2}{V(q^2) q^4} \left(Q_r - \frac{1}{3}\right)$$
(41)

where r=u, d refers to the spectator quark in the B meson. The corrections are tiny, at $q^2=15\,\mathrm{GeV}^2$ we have $\Delta a_{9,H6a}(K_{\parallel}^*)=\Delta a_{9,H6a}(K_{\perp}^*)=0.003$ for r=u, and $\Delta a_{9,H6a}(K_{\parallel}^*)=-0.001, \ \Delta a_{9,H6a}(K_{\perp}^*)=-0.006$ for r=d. This is again a consequence of the $1/q^2$ suppression at large values of q^2 , see also [30].

5 Large vs. small recoil energy of the kaon

The OPE for the correlator (10), applied to $B \to K^{(*)}l^+l^-$, is valid as long as the energy of the kaon in the B-meson rest frame

$$E_K = \frac{m_B^2 + m_K^2 - q^2}{2m_B} \tag{42}$$

is small compared to $\sqrt{q^2}$. This condition is certainly fulfilled in the vicinity of the endpoint, but even for q^2 as low as $15\,\mathrm{GeV^2}$, just above the narrow-resonance region, $E_K=1.24\,\mathrm{GeV}$ is still fairly small in comparison to the hard scale. On the other hand, such a value of E_K is already larger than the QCD scale Λ_{QCD} and one could consider using the factorization methods applicable to the case of energetic kaons. For q^2 at $15\,\mathrm{GeV^2}$, or somewhat above, we have a transition region for the applicability of factorization methods for large E_K and the OPE for small E_K . It is this transition we wish to explore in the present section.

QCD factorization (QCDF) at large kaon recoil requires $E_K \gg \Lambda_{QCD}$ for arbitrary q^2 . The OPE method requires $\sqrt{q^2} \gg E_K$, Λ_{QCD} . Both scenarios are consistent with the case

$$\sqrt{q^2} \gg E_K \gg \Lambda_{QCD}$$
 (43)

which can be realized, at least approximately, as we have seen above. In both, the QCDF and the OPE scenario, the leading contributions to the $B \to K^{(*)}l^+l^-$ amplitudes are given by short-distance quantities multiplying the standard $B \to K^{(*)}$ form factors. In QCDF, the first corrections to these form-factor terms come from hard-spectator interactions, which have been computed in [15]. Normalized to a_9 (called C_9 in [15]) these corrections may be written as $\Delta a_{9,\parallel+}^{(nf)} + \Delta a_{9,\parallel-}^{(nf)}$, where the indices refer to the notation of [15]. For definiteness we will consider the case of a pseudoscalar kaon in the following. Neglecting the small penguin coefficients C_3, \ldots, C_6 and adapting the expressions in [15] to our notation, the second contribution reads $(\bar{u} \equiv 1 - u)$

$$\Delta a_{9,\parallel^{-}}^{(nf)} = -\frac{\pi \alpha_{s} C_{F}}{2N} \frac{m_{b} f_{B} f_{K} Q_{q}}{m_{B} f_{+}(q^{2})} \int \frac{d\omega \, \phi_{-}(\omega)}{m_{B} \omega - q^{2} - i\epsilon}$$

$$\cdot \int_{0}^{1} du \, \phi_{K}(u) \left[\frac{8C_{8}}{\bar{u} + u \frac{q^{2}}{m_{B}^{2}}} + \frac{6m_{B}}{m_{b}} C_{1} h(z, \hat{s})|_{q^{2} \to \bar{u} m_{B}^{2} + uq^{2}} \right]$$

$$(44)$$

In the OPE limit, defined by treating $q^2 \sim m_B^2$ and taking (43), we have $\Delta a_{9,\parallel}^{(nf)} \sim 1/m_B^3$. This term is therefore subleading with respect to (31), consistent with the absence of a C_8 term in $\Delta a_{9,H5}(K)$. The remaining term is given by

$$\Delta a_{9,\parallel+}^{(nf)} = -\frac{\pi \alpha_s C_F}{2N} C_1 Q_c \frac{f_B f_K}{m_B f_+(q^2)} \int \frac{d\omega}{\omega} \phi_+(\omega) \int_0^1 du \, \phi_K(u) t_{\parallel} \tag{45}$$

where $(E = E_K)$

$$t_{\parallel} = \frac{2m_B}{\bar{u}E}I_1 + \frac{\bar{u}m_B^2 + uq^2}{\bar{u}^2E^2}(B_0(\bar{u}m_B^2 + uq^2) - B_0(q^2))$$
(46)

$$B_0(s) = \sqrt{1 - \frac{4m_c^2}{s}} \left(\ln \frac{1 - \sqrt{1 - \frac{4m_c^2}{s}}}{1 + \sqrt{1 - \frac{4m_c^2}{s}}} + i\pi \right)$$
(47)

$$I_1 = 1 + \frac{2m_c^2}{\bar{u}(m_B^2 - q^2)} (L_1(x_+) + L_1(x_-) - L_1(y_+) - L_1(y_-))$$
(48)

$$x_{\pm} = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{m_c^2}{\bar{u}m_B^2 + uq^2}\right)^{1/2} \pm i\epsilon, \qquad y_{\pm} = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{m_c^2}{q^2}\right)^{1/2} \pm i\epsilon$$
 (49)

$$L_1(w_{\pm}) = \text{Li}_2\left(\frac{w_{\pm}}{w_{\pm} - 1}\right) - \frac{\pi^2}{6} + \ln(1 - w_{\pm})\left(\ln\frac{1 - w_{\pm}}{w_{\pm}} \pm i\pi\right), \qquad w_{\pm} = x_{\pm}, \ y_{\pm} \ (50)$$

The formulas from [15] have been slightly rewritten here to make them more convenient for application in the high- q^2 region where $4m_c^2/q^2 < 1$. To make contact with the OPE result we eliminate q^2 from (46) using $q^2 = m_B^2 - 2m_B E$ and expand in E/m_B . The terms proportional to $1/E^2$ and 1/E cancel and we find

$$t_{\parallel} = f(x) + \mathcal{O}\left(\frac{E}{m_B}\right) \tag{51}$$

with the function f(x) from (25). In the limit $E_K \ll m_B$ the QCDF formula (45) then reduces to the OPE result (31). We note that the hard-spectator correction (45), which is a leading-power contribution in QCDF, becomes a power-correction $\sim 1/q^2 \sim 1/m_B^2$ in the OPE regime. This behaviour is related to the form factor $f_+(q^2)$, which scales as $1/m_B^{3/2}$ at small q^2 and as $m_B^{1/2}$ at large q^2 .

We may use the preceding comparison to check the validity of the OPE at the relatively low values of q^2 around 15 GeV². Assuming that the large-energy limit for the kaon is a reasonable approximation, the QCDF expression (45) and the OPE result (31) differ only by the replacement

$$\int_0^1 du \,\phi_K(u)t_{\parallel} = -5.05 + 2.70i \qquad \to \qquad f(x)\frac{m_B^2}{q^2} = -5.87 + 3.55i \qquad (52)$$

where the numerical values are obtained at $q^2=15\,\mathrm{GeV}^2$ for our standard set of parameters, neglecting the kaon mass for consistency. The real part of the OPE approximation is 16% larger in magnitude than the QCDF result, for the imaginary part the discrepancy is about 30%. Such differences are to be expected since the expansion leading from QCDF to the OPE is governed by $E_K/\sqrt{q^2}\approx 0.3$. Within this accuracy, the OPE formula still gives a very good estimate of the more complete QCDF result at $q^2=15\,\mathrm{GeV}^2$. For larger values of q^2 the numerical difference between the two sides in (52) becomes smaller and towards the endpoint the OPE is the more appropriate description.

The above exercise suggests that even at q^2 as low as $15 \,\text{GeV}^2$ the OPE is a valid method to obtain the second-order correction (31). The difference with the QCDF estimate is immaterial in view of the very small overall size of the effect.

The transition from large to small recoil energy has already been considered in [22] for the $B \to K$ form factor ratio f_T/f_+ . Also in this and in analogous cases, hard-spectator corrections, which are leading-power effects at large recoil, become power suppressed for soft kaons. Similar observations hold for weak annihilation [22].

6 Toy-model analysis of duality violation

The OPE defines a systematic framework to compute the correlator (10) at high q^2 in QCD. In the Minkowski region $q^2 > 0$, there will be uncertainties in the OPE-based predictions that go beyond those due to neglected orders in α_s or in Λ/m_b . Such effects are referred to as violations of quark-hadron duality. We investigate their importance first within a toy model for the charm loops in $B \to K l^+ l^-$. In section 7 the model is generalized to be closer to the realistic case.

6.1 Description of the model

It is illuminating to consider the systematics of duality for the charm-loop contribution to $B \to K l^+ l^-$ in the simplified context of the toy model introduced in [12]. This model assumes the existence of two leptons, l_1 with a large mass m_1 and l_2 with mass $m_2 = 0$, and the effective weak Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{G}{\sqrt{2}} \left[(\bar{l}_2 l_1)_{V-A} (\bar{c}c)_{V-A} - (\bar{l}_2 l_1)_{V-A} (\bar{t}t)_{V-A} \right]$$
 (53)

All particles have standard strong and electromagnetic interactions. Then \mathcal{H}_{eff} gives rise to a loop-induced process $l_1 \to l_2 \, e^+ e^-$ via charm- and top-quark penguin diagrams with a GIM-like cancellation between them. The hadronic physics is fully contained in current correlators of the form

$$\Pi_c^{\mu\nu} = i \int d^4x \, e^{iq \cdot x} \, \langle 0|T \, j_c^{\mu}(x) j_c^{\nu}(0)|0\rangle \equiv (q^{\mu}q^{\nu} - q^2g^{\mu\nu}) \, \Pi_c(q^2)$$
 (54)

with $j_c^{\mu} = \bar{c} \gamma^{\mu} c$. The decay amplitude can then be written as

$$A(l_1 \to l_2 e^+ e^-) = -\frac{G}{\sqrt{2}} e_c e^2 \Pi(q^2) \bar{l}_2 \gamma^{\mu} (1 - \gamma_5) l_1 \bar{e} \gamma_{\mu} e$$
 (55)

where $\Pi \equiv \Pi_c - \Pi_t$ is the difference of the charm and top contributions. We take $m_t > m_1$, and thus Im Π comes only from the charm sector. The correlator $\Pi(q^2)$ fulfills the dispersion relation

$$\Pi(q^2) - \Pi(0) = \frac{q^2}{\pi} \int_0^\infty \frac{dt}{t} \frac{\text{Im } \Pi(t)}{t - q^2 - i\epsilon}$$
 (56)

where the subtraction constant $\Pi(0)$ is fixed within the model and can be computed in perturbation theory. To leading order it reads

$$\Pi(0) \equiv \Pi_c(0) - \Pi_t(0) = \frac{N}{12\pi^2} \ln \frac{m_t^2}{m_c^2}$$
(57)

The form of (55) for the amplitude holds to lowest order in G and e^2 , but to all orders in the strong coupling. The decay $l_1 \to l_2 e^+ e^-$ in this model shares important similarities with $B \to K e^+ e^-$, but the hadronic dynamics is simplified to the physics of quark-current correlators. The role of resonances and quark-hadron duality can thus be illustrated in a transparent way.

From (55) we obtain the differential decay rate (with $s = q^2/m_1^2$)

$$\frac{d\Gamma(l_1 \to l_2 e^+ e^-)}{ds} = \frac{G^2 \alpha^2 m_1^5}{108\pi^5} (1 - s)^2 (1 + 2s) \left| C + \Delta(q^2) \right|^2$$
 (58)

where we defined

$$C \equiv 2\pi^2 \Pi(0) \tag{59}$$

and

$$\Delta(q^2) \equiv 2\pi^2 \left(\Pi(q^2) - \Pi(0) \right) \tag{60}$$

To lowest order we have

$$C = \ln \frac{m_t}{m_c} \tag{61}$$

and the partonic expression for $\Delta(q^2)$ in one-loop approximation is

$$\Delta_{q}(q^{2}) = \frac{5}{6} + \frac{x}{2} - \frac{1}{4} (2+x) \sqrt{|1-x|} \begin{cases} 2 \arctan \frac{1}{\sqrt{x-1}}; & x > 1\\ \ln \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} - i\pi; & x < 1 \end{cases}$$
 (62)

where $x = 4m_c^2/q^2$. For typical values of the parameters (e.g. $m_c = 1.4 \,\mathrm{GeV}$, $m_1 = m_b = 4.8 \,\mathrm{GeV}$, $m_t = 167 \,\mathrm{GeV}$) we have C = 4.78, whereas $\mathrm{Re}\Delta_q(q^2)$ first rises from 0 at $q^2 = 0$ to 4/3 at $q^2 = 4m_c^2$ and then drops again monotonically to the small negative value -0.07 at $q^2 = m_b^2$. The coefficient C represents the short-distance contribution of the amplitude. It is real and larger (parametrically as well as numerically) than the quark-level charm contribution $|\Delta_q|$.

The decay rate (58) is proportional to

$$|C + \Delta|^2 = C^2 + 2C\operatorname{Re}\Delta + |\Delta|^2 \tag{63}$$

As long as $|\Delta|$ is small compared to C, there is a clear hierarchy among the three terms on the r.h.s. of (63): The first, short-distance term C^2 dominates and the next term gives the correction to first-order in Δ , whereas the final term enters at second order.

These features are qualitatively similar in the case of $B \to K l^+ l^-$.

6.2 Shifman model for charm correlator

In order to investigate the systematics of duality violation in $l_1 \rightarrow l_2 e^+ e^-$, we find it convenient to consider first a simple model for the quark-current correlator, which has been proposed in [31,32,33]. In this model the correlator is represented as an infinite sum over resonances, which include finite width effects. In its original form it applies to massless quarks and we will correspondingly neglect the charm-quark mass in the present section. A detailed discussion of the model and its use in illustrating duality violation in the R ratio and similar quantities has been given in [31,32,33]. The model has also been used to study duality violation in τ decays in [34].

In Shifman's model the correlator Δ in (60) reads

$$\Delta(q^2) = -\frac{N}{6} \frac{1}{1 - b/\pi} \left[\psi(z+1) + \gamma \right]$$
 (64)

where $\psi(z) = d \ln \Gamma(z)/dz$ is the digamma function and

$$z = (-r - i\epsilon)^{1 - b/\pi} \quad \text{with} \quad r = \frac{q^2}{\lambda^2}$$
 (65)

N=3 is the number of colours, λ is a scale corresponding to the string tension in QCD and $b \equiv B/N = \Gamma_n/M_n$ is a (small) parameter related to the width-to-mass ratio of the resonances.

The model expression for Δ in (64) has the correct analytic behaviour (a cut for positive q^2 but no other singularities on the physical sheet) and it reproduces the asymptotic result of QCD in the limit of large q^2 ,

$$\Delta(q^2) \to -\frac{N}{6} \ln \frac{-q^2 - i\epsilon}{\lambda^2}$$
 (66)

Using the identity

$$\psi(z+1) + \gamma \equiv [\psi(-z) + \gamma - i\pi]_1 + [-\pi \cot \pi z + i\pi]_2$$
 (67)

the function in (64) can be decomposed into two parts, $\Delta = \Delta_1 + \Delta_2$, corresponding to the two brackets in (67). Δ_2 is an oscillating function of q^2 , exponentially suppressed for large q^2 . It represents the duality violating component of Δ . The function Δ_1 is a monotonous function, which gives the OPE approximation to Δ . The real part of these two functions is shown in Fig. 4. A plot of the imaginary part of Δ can be found in [32,33].

The duality violating part of Re Δ can be approximated as

$$\operatorname{Re}\Delta_{2}(q^{2}) = -\frac{N}{6} \frac{1}{1 - b/\pi} \operatorname{Re}\left[-\pi \cot \pi z + i\pi\right] \approx -\frac{N\pi}{3} \exp(-2\pi br) \sin(2\pi r)$$
 (68)

if $2\pi br \gg 1$ and $b \ln r/\pi \ll 1$.

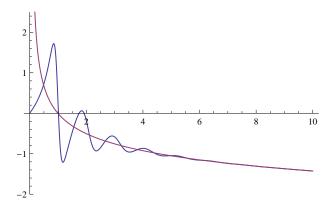


Figure 4: Shifman model for charm loop: Re $\Delta(q^2)$ as a function of q^2/λ^2 for $b \equiv B/N = 1/6$. The true function (oscillating curve) is compared with the OPE approximation.

In the decay rate integrated over the high- q^2 part of the spectrum, the duality violating contribution enters proportional to

$$\int_{s_0}^1 ds \, (1+2s)(1-s)^2 \operatorname{Re}\Delta_2 \approx -\frac{N\pi}{3} \int_{s_0}^1 ds \, (1+2s)(1-s)^2 \, \exp(-2\pi bus) \sin(2\pi us)$$

$$= -\frac{N}{6}(1+2s_0)(1-s_0)^2 \frac{1}{u} \exp(-2\pi b s_0 u) \cos(2\pi s_0 u) + \mathcal{O}\left(\frac{b}{u}, \frac{1}{u^2}\right)$$
(69)

where

$$s = q^2/m_1^2$$
 $u = m_1^2/\lambda^2$ $r = us$ (70)

In (69) we have used the approximation from (68). Typical values of the parameters are

$$b = \frac{1}{6} \qquad u = 10 \tag{71}$$

The value of u = 10 corresponds for instance to $\lambda^2 = 2.3 \,\text{GeV}^2$ and $m_1^2 = 23 \,\text{GeV}^2$. The quantity in (69) is shown in Fig. 5 as a function of s_0 .

We comment on several important aspects of these results.

- The duality violating component of Re Δ in (68) exhibits the characteristic oscillating behaviour in $r = q^2/\lambda^2$ with an exponential suppression governed by br. The analogous expression for the duality violating term in Im Δ , which has a cosine instead of the sine, has been given in [32,33].
- Eq. (69) displays the duality violating contribution from Re Δ to the partially integrated decay rate. The integration over s extends from a suitably chosen lower limit s_0 up to the end of the spectrum. The parameter s_0 should be large enough such that the OPE still remains reasonable at the corresponding value of q^2 . Using the approximation in (68) and expanding in the small quantities b and 1/u, we find

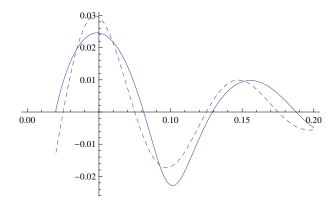


Figure 5: Shifman model for charm loop: Duality violating contribution $\int_{s_0}^1 ds \, (1+2s)(1-s)^2 \text{Re } \Delta_2$ as a function of s_0 with parameters b=1/6 and u=10. The full result (solid curve) is compared with the approximation given as the last term in eq. (69) (dashed curve).

the explicit result written as the last expression in (69). This expression is an oscillating function of s_0 , with frequency $2\pi u$, multiplied by an exponential suppression factor. The latter is active when the exponent is large, at least $2\pi b s_0 u \sim 1$. In addition, the entire term is further suppressed by one power of $1/u = \lambda^2/m_1^2$. This effect remains even if the exponential suppression is not fully developed. The power suppression arises because the oscillating contributions average out in the integral, except for a remainder $\sim 1/u$ near the lower end of integration s_0 .

• The duality violating term (69) is plotted in Fig. 5 for a semi-realistic choice of parameters, where $m_1 = 4.8 \,\text{GeV}$ and $\lambda^2 = 2.3 \,\text{GeV}^2$. The smallest values of s_0 shown, $s_0 \sim 0.05$, correspond to $q^2 \sim 1.2 \,\text{GeV}^2$ and $2\pi b s_0 u \sim 0.5$, which is already on the low side of the allowed range. From Fig. 5 we observe, first, that in the scenario considered here the simple approximation to (69) agrees very well with the full result. Second, the numerical size of the duality violating term is about ± 0.02 for the relatively low values of s_0 around 0.1. The effect diminishes quickly for larger s_0 due to the exponential suppression. The variation ± 0.02 from (69) amounts to $\pm 6.5\%$ when compared with the corresponding OPE expression

$$\int_{0.1}^{1} ds \, (1+2s)(1-s)^2 \operatorname{Re} \Delta_1 = -0.31$$
 (72)

We next turn to a discussion of the $|\Delta|^2$ term in (63), where the situation is systematically different from the case of Re Δ . The $|\Delta|^2$ term receives a contribution from the

duality violating component given by

$$\int_{s_0}^1 ds \, (1+2s)(1-s)^2 \, |\Delta_2|^2 \approx \int_{s_0}^1 ds \, (1+2s)(1-s)^2 \, \left(\frac{N\pi}{3}\right)^2 \exp(-4\pi b u s)$$

$$= \left(\frac{N\pi}{3}\right)^2 \frac{(1+2s_0)(1-s_0)^2}{4\pi b u} \exp(-4\pi b s_0 u) + \mathcal{O}\left(\frac{1}{(bu)^2}\right)$$
(73)

where in the second step an approximation similar to the one in (68) has been used. There is still an exponential suppression, which makes the entire term negligible for sufficiently large bs_0u . On the other hand, the power suppression with 1/u observed in (69) is softened into a behaviour as 1/(bu). For small b (comparable to 1/u or smaller) the violation of duality may become large. This is in qualitative agreement with the discussion of duality violation for the squared correlator $|\Pi(q^2)|^2$ given in [12]. The enhanced impact of duality violation is related to the absence of oscillations with alternating sign in the integrand of (73). We conclude that the $|\Delta|^2$ term in the integrated rate is particularly susceptible to violations of quark-hadron duality, which may lead to substantial (positive) deviations from the OPE result, unless q^2 is large enough for a strong exponential suppression of the effect. However, the uncertainties in the $|\Delta|^2$ term may be immaterial if this contribution is only a small correction to the dominant C^2 part in (63).

Let us finally illustrate the relative importance of the various contributions to the rate of $l_1 \to l_2 e^+ e^-$ using the Shifman model for the charm loop with the numerical input defined above. In the limit $m_c \to 0$ considered here the function C from (61) is modified to

$$C = \ln \frac{m_t}{\lambda} + \frac{\gamma}{2} + \frac{5}{6} = 5.82 \tag{74}$$

The relative contributions to the decay rate (58), (63) then read

$$\int_{0.1}^{1} ds \, g(s) \, C^2 = 13.60 \tag{75}$$

$$\int_{0.1}^{1} ds \, g(s) \, 2C \operatorname{Re} \Delta_1 = -3.59 \qquad [\pm 0.23] \tag{76}$$

$$\int_{0.1}^{1} ds \, g(s) \, |\Delta_1|^2 = 1.25 \qquad [+0.10] \tag{77}$$

with $g(s) = (1+2s)(1-s)^2$. The central values in (76) and (77) are based on the OPE result for Δ , the square brackets indicate the impact of duality violation. The uncertainty in (76) gives the variation due to $\text{Re}\Delta_2$. It has the relative size of $\pm 6.5\%$, which is reduced to $\pm 2\%$ in the sum of all contributions. The shift from duality violation in (77) is positive and essentially negligible in the present example.

6.3 Model for charm correlator based on BES data

In order to obtain a more realistic picture of the $c\bar{c}$ - spectrum, we are going to fit the BES data [35,36,37] for the R-ratio in the $c\bar{c}$ -region to a modified version of Shifman's

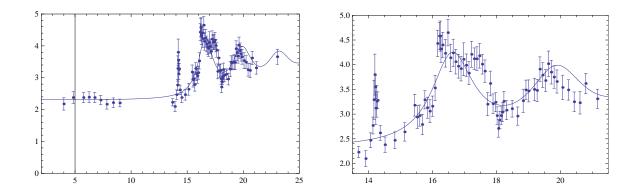


Figure 6: Simple fit to BES data for R vs. q^2/GeV^2 . Right: Detailed View.

model. The spectra of $c\bar{c}$ mesons can be accounted for by linear relations for the squared masses, $M_n^2 = n\lambda^2 + M_0^2$, $n = 1, 2, 3, \ldots$, similarly to the case of the light mesons [38]. The trajectory of the n^3S_1 charmonia, the $J^{PC} = 1^{--}$ states $\psi(3097)$, $\psi(3686)$, $\psi(4040)$, $\psi(4415)$, ..., for instance, follows this pattern. Starting form the third resonance (n = 3), these states can decay into open charm and have widths of the order of Λ_{QCD} . The first two are extremely narrow and may be described separately, but their properties are unimportant for duality violation, which is related to the infinite tower of high-n resonances. We therefore choose an ansatz where the sum over resonances begins at n = 3 rather than n = 1. A finite width is included in analogy to (65) and the variable q^2 is shifted by a constant into $q^2 - 4m_c^2$. This leads to the following expression for the imaginary part of the correlator or, equivalently, the R ratio

$$R = R_{\text{light}} - \frac{4}{3} \frac{1}{(1 - b/\pi)\pi} \operatorname{Im} \psi(3 + z), \qquad z = \left(-\frac{q^2 - 4m_c^2 + i\epsilon}{\lambda^2}\right)^{1 - b/\pi}, \tag{78}$$

The individual resonances are located at $q^2 = n\lambda^2 + 4m_c^2$ (n = 3, 4, 5, ...) in the limit $b \to 0$. We observe that a rough description of the BES data [35] can already be obtained with this formula, where we find $R_{\text{light}} = 2.31$ from the measured R-ratio below charm threshold, $m_c = 1.33$ GeV, $b \simeq 0.082$ and $\lambda^2 \simeq 3.08$ GeV². This yields the result shown in Fig. 6, corresponding to a $\chi^2/\text{d.o.f.} \simeq 2.5$. We remark that our values for the fit parameters m_c and λ^2 are in agreement with the results of [38]. There is a second trajectory of 1⁻⁻ charmonia, the n^3D_1 states. Of these the first two resonances $\psi(3770)$ and $\psi(4160)$ are known. The first one is barely above threshold and still rather narrow. It may be considered separately, similar to $\psi(3097)$ and $\psi(3686)$. Note also that $\psi(3770)$ is still below our default choice for the lower limit of the high- q^2 region, $q^2 \ge 15$ GeV². The remaining resonances n^3D_1 are rather close to the resonances $(n+1)^3S_1$ for $n \ge 2$. For an approximate treatment it appears justified to subsume such a pair of close resonances under a single peak and keep the ansatz given in (78). The accuracy of this description can be gauged by inspecting Fig. 6. In any case, the normalization of the second term of R in (78) is fixed in the large- q^2 limit by the free-quark result.

The fit could be refined by including the low-lying, narrow resonances in the fit ansatz. Any finite number of resonances does not change the asymptotic behaviour responsible for duality violation. Also the fit parameters are not much affected by such modifications. For example, including the $\psi(3770)$ reduces the $\chi^2/\text{d.o.f.} \rightarrow 1.7$, while the parameters of the continuum ansatz remain almost unchanged, $R_{\text{light}} = 2.26$, $m_c = 1.33 \text{ GeV}$, $b \simeq 0.078$ and $\lambda^2 \simeq 3.08 \text{ GeV}^2$. We will therefore be content with the simple representation given in (78) above.

7 Quark-hadron duality in $B \to K l^+ l^-$

7.1 General considerations

The analytic structure of the matrix element of the operator product in (10) can be infered from

$$-\frac{q^{2}}{8\pi^{2}}\langle \mathcal{K}_{H}^{\mu}\rangle = i \int d^{4}x \, e^{iq\cdot x} \, \langle \bar{K}(k)|T \, j^{\mu}(x)H^{c}(0)|\bar{B}(p)\rangle$$

$$= \sum_{X} \frac{(2\pi)^{3}\delta(\vec{q} + \vec{k} - \vec{p}_{X})}{p_{X0} - q_{0} - k_{0} - i\epsilon} \langle \bar{K}(k)|j^{\mu}(0)|X\rangle \langle X|H^{c}(0)|\bar{B}(p)\rangle$$

$$+ \sum_{X} \frac{(2\pi)^{3}\delta(\vec{q} + \vec{p}_{X} - \vec{p})}{p_{X0} + q_{0} - p_{0} - i\epsilon} \langle \bar{K}(k)|H^{c}(0)|X\rangle \langle X|j^{\mu}(0)|\bar{B}(p)\rangle$$
 (79)

In order to discuss the properties of the matrix element in (79) we make the following simplifications. First, we neglect the small penguin contributions in the weak Hamiltonian, that is we take

$$H^c = C_1 Q_1^c + C_2 Q_2^c (80)$$

Second, in the electromagnetic current we retain only the charm-quark component, $j^{\mu} = Q_c \bar{c} \gamma^{\mu} c$, and consider only contributions where the charm fields in H^c are contracted with those in j^{μ} . This neglects terms where the charm-anticharm pair from H^c annihilates into gluons before connecting to the electromagnetic current. Such contributions are of higher order in α_s and not essential for the problem we want to address.

We consider the matrix element in (79) as a function of q_0 , keeping the kaon energy k_0 fixed at a value of order 1 GeV. This can be achieved by injecting a spurion 4-momentum [39] $r = (r_0, 0, 0, 0)$ into the H^c vertex. Then p+r=q+k and for p_0 and k_0 fixed at their physical values the variable $r_0 = q_0 + k_0 - p_0$ grows with q_0 . The physical kinematics is recovered for r = 0. Under the simplifying assumptions specified above, the intermediate state X in the first sum in (79) always contains a $c\bar{c}$ pair and a strange quark, in general together with other hadronic states, and thus $p_{X0} - k_0 \gtrsim 2m_c$. The state X in the second sum contains a $c\bar{c}$ pair and a $p_{X0} - p_0 \gtrsim 2m_c$. The matrix element in (79) is then seen to be an analytic function of q_0 in the entire q_0 -plane, except for two branch cuts at $2m_c \lesssim q_0 < \infty$ and at $-\infty < q_0 \lesssim -2m_c$. If we would relax the simplifications

above and allowed for intermediate states X without $c\bar{c}$ pairs, the cuts would extend down to lower values of $|q_0|$ on the real axis.

The OPE of the matrix element in (79) can be justified for q_0 on the imaginary axis, sufficiently far from the origin, that is at $q_0 = iq_{0E}$, for $q_{0E} \gg \Lambda_{QCD}$. The OPE defined in this way in the Euclidean can then be analytically continued, term by term, from imaginary q_0 onto the positive real axis, corresponding to the Minkowskian domain. Terms that are exponentially suppressed in Λ_{QCD}/q_{0E} for large positive q_{0E} become oscillating functions of q_0 in the Minkowskian case, that is for large positive q_0 [39]. These oscillating terms are invisible at any finite order in the OPE and represent the duality violating contribution.

7.2 Quantitative estimate of duality violation

For a quantitative estimate of duality violation we have to resort to a model of the hadronic correlator in (79). To this end we write the Hamiltonian in (80) as

$$H^{c} = a_{2} (\bar{s}b)_{V-A} (\bar{c}c)_{V-A}$$
(81)

and assume a factorization of the currents, that is we neglect interactions between $\bar{c}c$ and the $\bar{B} \to \bar{K}$ system. The coefficient a_2 is then treated as a phenomenological parameter. With these simplifications the correlator in (79) reduces to

$$\langle \mathcal{K}_H^{\mu} \rangle = \frac{16\pi^2}{3} a_2 \langle (\bar{s}b)_{V-A} \rangle^{\mu} \Pi_c(q^2)$$
 (82)

where we omitted the longitudinal component $\sim q^{\mu}$. Π_c is the current correlator defined in (54). The charm loop in (82) contributes to the coefficient a_9 in the amplitude of $\bar{B} \to \bar{K} l^+ l^-$ a term

$$\Delta a_9 = a_2 d, \qquad d \equiv \frac{16\pi^2}{3} \left(\Pi_c(q^2) - \Pi_c(0) \right)$$
 (83)

In the model of section 6.3 we have

$$d = -\frac{4}{3} \frac{1}{1 - b/\pi} \left[\psi(z+3) - \psi(z_0+3) \right]$$
 (84)

where

$$z = (-r - i\epsilon)^{1-b/\pi}, \quad r = \frac{q^2 - 4m_c^2}{\lambda^2} \equiv u(s - s_c), \quad u = m_B^2/\lambda^2$$
 (85)

and $z_0 = z(q^2 = 0)$. For the parameters we use the following values

$$\lambda^2 = 3.08 \,\text{GeV}^2, \quad m_c = 1.33 \,\text{GeV}, \quad b = 0.082$$
 (86)

We will not employ the model (83) to describe the entire charm-loop contribution, but only to estimate its duality violating component. The remainder is more reliably obtained by the OPE itself. In order to extract the term that represents duality violation within the model, we decompose

$$\psi(z+3) - \psi(z_0+3) \equiv [\psi(-z-2) - \psi(z_0+3) - i\pi]_1 + [-\pi \cot \pi z + i\pi]_2$$
 (87)

This decomposition is useful for $q^2 \gtrsim 15\,\mathrm{GeV}^2$, when the second term starts being exponentially suppressed and gives the duality violating contribution. In correspondence with (87) we have $d \equiv d_1 + d_2$ and the duality violating term is

$$d_2 = -\frac{4}{3} \frac{1}{1 - b/\pi} \left[-\pi \cot \pi z + i\pi \right]_2 \approx -\frac{8\pi}{3} \exp(-2\pi br) \left(\sin 2\pi r - i \cos 2\pi r \right)$$
 (88)

When integrating the $|a_9|^2$ part of the $\bar{B} \to \bar{K} l^+ l^-$ rate over the high- q^2 region, from a lower limit $q_0^2 = s_0 m_B^2$ to the end of the spectrum, the relative size of the duality violating effect is then given by

$$R_{DV,1} = \frac{2a_2}{a_9} \frac{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s) \operatorname{Re} d_2}{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s)}$$
(89)

to first order in the charm-loop contribution. Here we have assumed a_2 to be real. Using the approximation in (88) and proceeding as in (69) we find

$$|R_{DV,1}| \lesssim \frac{8}{3} \frac{a_2}{a_9} \frac{\lambda_K^{3/2}(s_0) f_+^2(s_0)}{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s)} \frac{1}{u} \exp(-2\pi b u(s_0 - s_c)) \tag{90}$$

This formula gives an excellent approximation of the full result based on (89). We note the (mild) exponential suppression and the power suppression by $1/u = \lambda^2/m_B^2$. For a_2 we take the value $a_2 = 0.3$, which is large enough to reproduce the measured $B \to K\psi$ branching fraction within the factorization ansatz. We recall that $a_9 \approx 4$. Then, for $0.5 < s_0 < 0.6$, that is for q_0^2 in the vicinity of 15 GeV^2 , the relative correction $|R_{DV,1}|$ is below 3%. In the rate for $\bar{B} \to \bar{K}l^+l^-$, $|a_{10}|^2$ is added to $|a_9|^2$, which roughly doubles the result. The net effect of the uncertainty from (90) for the rate integrated over the high- q^2 region is therefore only about 1.5%.

As discussed in section 6, the second order effect in d_2 is qualitatively different. It has no cancellations due to oscillating terms, giving a positive correction, and its impact increases with decreasing b. If local duality is at least roughly fulfilled, as is the case for high enough $q^2 \gtrsim 15 \,\text{GeV}^2$, the duality violation from $|d_2|^2$ is still suppressed, being of second order in the small quantity a_2/a_9 . The relative size of this component is

$$R_{DV,2} = \frac{a_2^2}{a_9^2} \frac{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s) \, |d_2|^2}{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s)}$$

$$\approx \left(\frac{8\pi a_2}{3a_9}\right)^2 \frac{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s) \, \exp(-4\pi bu(s-s_c))}{\int_{s_0}^{s_m} ds \, \lambda_K^{3/2}(s) f_+^2(s)}$$
(91)

where the second term uses the approximation in (88). The approximate form and the full result in (91) agree reasonably well. With our set of parameters we find $R_{DV,2} = 0.015$ at $q_0^2 = 15 \,\text{GeV}^2$, about half the size of $|R_{DV,1}|$.

The phenomenological value $a_2 = 0.3$ used above is larger than the perturbative result for the coefficient a_2 . Thus it effectively absorbs the (partly unknown) effects from factorizable and non-factorizable corrections. We should emphasize here that the analytic structure of the latter is, in general, more complicated than it is implied by the approximation to $\langle \mathcal{K}_H^{\mu} \rangle$ in (82). Globally using the larger value of a_2 in our numerical estimate of duality violation thus corresponds to the pessimistic scenario where the oscillating terms from the non-factorizable corrections to the charm-loop are added coherently. In reality, we expect that at least some destructive interference between the various contributions appears, and the amount of duality violation should even be smaller than our estimate.

Our model for the charm-loop in (82) is similar to the ansatz originally proposed in [40] and used since then in many phenomenological studies. However, our motivation for considering this model is essentially different. In contrast to [40] it is not our goal to model the hadronic effects on the spectrum point by point in the q^2 distribution. More relevant than the detailed shape of the spectrum is the rate integrated over the entire high- q^2 region, which is best described in a model-independent way by the OPE, as mentioned above. We rather employ the model to get an indication of the duality violating effects, which are not captured in an OPE calculation. For this purpose the Shifman model for the current correlator is adapted to the charm-quark case Π_c , with a choice of parameters consistent with the basic features of the most recent experimental data from BES. Note that the Krüger-Sehgal ansatz for Π_c [40], consisting of a spectral function with a finite number of resonances and a flat continuum for large q^2 , contains no information on duality violation. The model we use to estimate duality violation is very simple and involves many assumptions. Still we expect it to indicate the systematics and the typical size of the effect, which presumably is closely connected with the $c\bar{c}$ resonance structure in the charm-loop contribution.

8 Comments on the literature

The high- q^2 region of $B \to K^*l^+l^-$ has also been analyzed in the framework of an OPE in [18] and a recent application was presented in [41]. Our approach differs from the analysis of [18] in several respects. We go beyond the work of [18] by addressing in detail the issue of duality violation and the basis of the OPE formalism. In addition, we extend the OPE to include second-order power corrections, which we estimate quantitatively. A list of new results is given in the Conclusions. Here we would like to comment further on two conceptual differences between [18] and our formulation. This concerns, first, the construction of operators in the OPE and, second, the treatment of charm quarks.

The operators in our approach are built from b-quark fields in full QCD rather than using HQET. This is convenient because the operator basis is simpler and the OPE becomes particularly transparent. Another advantage is that the matrix elements of the

leading operators are given by the usual form factors in full QCD. Unlike in a HQET framework, not all dependence on m_b is made explicit, but this is not essential from a practical point of view and still allows the consistent inclusion of power corrections to a given order in the expansion. This approach has been used for instance in the OPE for inclusive B decays applied to the computation of the lifetime difference of B_s mesons [42,43]. These investigations included power corrections [42] as well as corrections of $\mathcal{O}(\alpha_s)$ [43]. By contrast, in [18] a matching onto HQET operators is performed from the start. This leads to a proliferation of operators, whose matrix elements are not given by the usual form factors. In fact, to reach a simplification of the resulting HQET expressions, the authors of [18] partly undo the matching, from HQET back to full QCD expressions. We prefer to employ the full-QCD formulation throughout, in view of the advantages mentioned above.

We next turn to the second point, the treatment of charm. The authors of [18] perform an explicit expansion in m_c/m_b , which corresponds to assuming the hierarchy $m_c \ll m_b$. This implies that the charm quark is not integrated out at the scale $m_b \approx \sqrt{q^2}$ and continues to be an active field below this scale. Operators with charm-quark fields are then present in the OPE.

We will argue that it is conceptually simpler to integrate out charm immediately at the scale m_b , assuming a hierarchy $m_b \sim m_c \gg \Lambda_{QCD}$, and that this can be done without any loss in accuracy. Charm-quark effects are then entirely contained in the coefficients of the operators, as is apparent from the formulation of the OPE given in section 3.

To illustrate the point we consider the following example. Let us take the scenario where $m_c \ll m_b$. In this case the operator coefficients should be evaluated with $m_c = 0$. On the other hand, additional operators involving charm would have to be included in the OPE. With explicit $c\bar{c}$ fields, such operators arise at dimension 6 or higher. For instance, radiating the virtual photon from the two charm lines in the effective Hamiltonian (80) and leaving all four quark lines open, gives the dimension-6 operator

$$\mathcal{K}_{H6c}^{\mu} = 16\pi^2 Q_c \frac{q_{\lambda}}{q^4} \left[\bar{c}_i \gamma^{\mu} (1 - \gamma_5) b_j \, \bar{s}_k \gamma^{\lambda} (1 - \gamma_5) c_l - \{ \mu \leftrightarrow \lambda \} \right] \left(\delta_{ij} \delta_{kl} C_1 + \delta_{il} \delta_{kj} C_2 \right) \tag{92}$$

in analogy to (26). This operator has $\bar{B} \to \bar{K}^{(*)}$ matrix elements contributing to the decay amplitude. Assuming next that $m_c \gg \Lambda_{QCD}$, the contribution (92) can be simplified by integrating out the charm fields in a further step. Contracting the charm lines into a loop, to which a gluon field is attached, induces below m_c the dimension-6 operator

$$\mathcal{K}^{\mu}_{H6d} = \frac{4}{3} C_1 Q_c \ln \frac{\mu^2}{m_c^2} \frac{q_{\lambda}}{q^4} i \varepsilon^{\lambda \mu \alpha \beta} \, \bar{s} \gamma_{\alpha} (1 - \gamma_5) g D^{\nu} G_{\nu \beta} b \tag{93}$$

The coefficient of this operator reproduces the logarithmic m_c -dependence at this order in the OPE. A similar discussion has been presented in [18], illustrating how the logarithmic term $m_c^4 \ln m_c^2$ in the coefficient of the leading dimension-3 operator (22) is recovered in an effective theory with four-quark operators of the type $(\bar{s}b)(\bar{c}c)$. Whereas the latter effect vanishes for $m_c \to 0$, we note that the coefficient in (93) has a logarithmic divergence

in this limit. This is of no consequence if $m_c \gg \Lambda_{QCD}$, where m_c still represents a hard scale. In fact, for higher-dimensional operators generated by (92), that is operators with more factors of the gluon field and its derivatives, the coefficients will scale as inverse powers of m_c . A similar situation exists for inclusive semi-leptonic $b \to c$ decays, where four-quark operators with charm, analogous to (92), also appear at third order in the OPE for $m_c \ll m_b$. The effect of the corresponding charm loops has been referred to as 'intrinsic charm' in [44,45,46], where the issue was discussed in great detail. The above consideration indicates how the nonperturbative $\bar{B} \to \bar{K}^*$ matrix element of operators such as (92) can be treated in an expansion in Λ_{QCD}/m_c . However, as demonstrated in [44,45,46] for semi-leptonic $b \to c$ decays, the same effects are also described in a framework where charm is integrated out at the m_b scale. No active charm fields need to be considered in this case and (92) is absent from the OPE. A difference between the OPE with or without active charm is that the framework with charm fields and with a strong hierarchy $m_c \ll m_b$ assumed, would offer the possibility of efficiently resumming logarithmic terms $\ln m_b/m_c$. Since such logarithms are not very large and in view of the additional power suppression of such terms, such resummations appear not to be necessary in practice. We also stress that effects such as (92), irrespective of their detailed treatment, are suppressed at least as $1/m_b^3$. Small differences in the method of their calculation are therefore hardly relevant. We thus conclude that integrating out charm at the scale m_b is entirely justified. The OPE is then constructed in a single step and with a simpler operator basis. For these reasons the approach appears preferable and we have adopted it here.

9 Conclusions

The amplitude for $\bar{B} \to \bar{M}l^+l^-$, $\bar{M} = \bar{K}, \bar{K}^*, \ldots$, has the general form given in (8). It contains the component

$$A_9^{\mu} = C_9 \langle \bar{M} | \bar{s} \gamma^{\mu} (1 - \gamma_5) b | \bar{B} \rangle + C_7 \frac{2im_b}{q^2} q_{\lambda} \langle \bar{M} | \bar{s} \sigma^{\lambda \mu} (1 + \gamma_5) b | \bar{B} \rangle + \langle \bar{M} | \mathcal{K}_H^{\mu} (q) | \bar{B} \rangle$$
 (94)

$$\mathcal{K}_{H}^{\mu}(q) = -\frac{8\pi^{2}}{q^{2}} i \int d^{4}x \, e^{iq \cdot x} \, T \, j^{\mu}(x) H^{c}(0)$$
(95)

which receives a nonlocal, hadronic contribution $\langle \mathcal{K}_H^{\mu} \rangle \equiv \langle \bar{M} | \mathcal{K}_H^{\mu} | \bar{B} \rangle$ from the matrix element of the nonleptonic weak Hamiltonian H^c in addition to the semileptonic form factor terms ($\sim C_9, C_7$). Although the hadronic part $\langle \mathcal{K}_H^{\mu} \rangle$ is relatively small numerically outside the narrow-resonance region ($\sim 10\%$ of A_9), it needs to be reliably computed in order to achieve very accurate predictions. We have presented a detailed study of the hadronic contribution in the region of large dilepton invariant mass $q^2 \gtrsim 15 \,\text{GeV}^2$, based on an operator product expansion in inverse powers of the hard scale $\sqrt{q^2}$. Working with b-quark fields in full QCD and factorizing the dependence on m_b , $\sqrt{q^2}$ and m_c into the coefficient functions, we obtain the following results:

- To leading order in the OPE and to all orders in α_s , $\langle \mathcal{K}_H^{\mu} \rangle$ is expressed in terms of the standard form factors parametrizing the matrix elements $\langle \bar{s} \gamma^{\mu} (1 \gamma_5) b \rangle$ and $\langle \bar{s} \sigma^{\lambda \mu} (1 + \gamma_5) b \rangle$ of dimension-3 operators, up to coefficients calculable in perturbation theory. To lowest order in α_s only $\langle \bar{s} \gamma^{\mu} (1 \gamma_5) b \rangle$ is present.
- In the chiral limit $(m_s = 0)$, the first power corrections appear only at second order $(\sim 1/q^2)$ and are governed by dimension-5 operators with a gluon field of the form $\bar{s}gG\Gamma b$. The corrections are computed explicitly, using the limit $E_K \gg \Lambda_{QCD}$ for the hadronic matrix elements, and are shown to be smaller than 1% for A_9 .
- For $m_s \neq 0$ the dimension-4 operators $m_s \bar{s} \gamma^{\mu} (1 + \gamma_5) b$ and $m_s \bar{s} \sigma^{\lambda \mu} (1 \gamma_5) b$, with right-handed strange quarks, can arise in the OPE. Because they are absent at order α_s^0 , their contribution will be suppressed to the negligible level of $\alpha_s m_s / m_b \sim 0.5\%$. Besides, no new form factors will be introduced by these operators.
- Within the OPE framework the effect of weak annihilation is a natural ingredient, which we have briefly discussed, mainly for illustration. Since in addition to being a third-order power correction it comes with small Wilson coefficients, its numerical impact of a few permille is entirely negligible.
- We have clarified the relationship between the OPE at high q^2 and QCD factorization at low q^2 by showing that both descriptions yield consistent results at intermediate values of $q^2 \approx 15 \,\text{GeV}^2$.
- A relevant topic at high q^2 is the issue of quark-hadron duality, which is closely related to the existence of an OPE. We have defined the OPE with the help of a spurion momentum that allows for an independent scaling of q^2 at fixed m_B and kaon energy. This allows one to clarify the analytic structure of the matrix elements of the correlator (95). We then employed a model based on an infinite series of charm resonances to estimate quantitatively the amount of duality violation, resulting in about $\pm 2\%$ for the rate integrated over the high- q^2 region. An important aspect is the different sensitivity to duality violation of the contributions to the rate linear or quadratic in $\langle \mathcal{K}_H^\mu \rangle$. The quadratic term is more vulnerable to duality violation, but numerically suppressed as a term of second order in the small charm-loop contribution. The systematics of duality have further been studied in a toy model for the rare decays, in which the factorization of the charm loop is exact.

The main conclusion is that the high- q^2 region of $B \to K^{(*)}l^+l^-$ is theoretically under excellent control. Decay rates and distributions are perturbatively calculable up to the nonperturbative effects accounted for by the standard form factors in full QCD. Further nonperturbative corrections are strongly suppressed and negligible within the accuracy of a few percent. The existence of the OPE implies that at high q^2 the theory of $B \to K^{(*)}l^+l^-$ has an even more solid basis than at low q^2 . An example for applications is the combined analysis of $B \to Kl^+l^-$ and $B \to K\nu\bar{\nu}$ [22,47]. Here the form factor

dependence can be essentially eliminated, which leads to precision observables sensitive to new physics effects [22]. Beyond $B \to K^{(*)}l^+l^-$, the results of our analysis apply to the high- q^2 region of exclusive rare decays with similar final states such as $B \to K\pi l^+l^-$, $B_s \to \phi l^+l^-$ and, with appropriate modifications, to $B \to \rho l^+l^-$ or $B \to \pi l^+l^-$.

A Complete basis of operators through dimension 4

We show that the OPE of \mathcal{K}^{μ}_{H} in (10) can be expressed in terms of

$$\mathcal{O}_{3,1}^{\mu} = \left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2}\right) \bar{s}\gamma_{\nu}(1 - \gamma_5)b , \qquad \mathcal{O}_{3,2}^{\mu} = \frac{im_b}{q^2} q_{\lambda} \bar{s}\sigma^{\lambda\mu}(1 + \gamma_5)b$$
 (96)

at the level of dimension-3 and dimension-4 operators in full QCD, in the chiral limit $m_s = 0$. For $m_s \neq 0$, $m_s \ll m_b$, right-handed strange quarks can also appear, together with a suppression factor of m_s/m_b . This leads to the two additional operators $\mathcal{O}_{4,1}^{\mu}$ and $\mathcal{O}_{4,2}^{\mu}$ in (17) and (18), which can be counted as operators of dimension 4. The matrix elements of $\mathcal{O}_{3,1}^{\mu}$, $\mathcal{O}_{3,2}^{\mu}$, $\mathcal{O}_{4,1}^{\mu}$ and $\mathcal{O}_{4,2}^{\mu}$ between \bar{B} and $\bar{K}^{(*)}$ are all given by the standard $\bar{B} \to \bar{K}^{(*)}$ form factors.

Together this implies that at leading (dimension 3) and next-to-leading order (dimension 4) in the OPE, and to all orders in α_s , standard form factors are the only hadronic matrix elements required. Corrections to these terms arise only at second order ($\sim \Lambda^2/q^2$) in the OPE through operators of dimension 5. In other words, there are no first-order power corrections ($\sim \Lambda/m_b$) in the OPE of \mathcal{K}_H^{μ} , except for the purely kinematical dependence on q^2 and m_b^2 in the coefficient functions. These functions contain first-order terms such as $(m_b^2 - q^2)/m_b^2$, which however are calculable and do not introduce unknown hadronic matrix elements.

We demonstrate the completeness of the basis $\mathcal{O}_{3,1}^{\mu}$, $\mathcal{O}_{3,2}^{\mu}$, $\mathcal{O}_{4,1}^{\mu}$ and $\mathcal{O}_{4,2}^{\mu}$ for operators of dimension 3 and 4 by enumerating all possibilities consistent with the relevant symmetries, and using the equations of motion.

We first assume $m_s = 0$. Operators of dimension 3 built from \bar{s}_L and b have the form

$$(\bar{s}_L \Gamma b)^{\mu}$$
, $\Gamma = 1, \gamma^{\alpha}, \sigma^{\alpha\beta}$ (97)

where Lorentz indices can be contracted with the metric g, the ε -tensor and factors of q to yield a 4-vector with index μ . Useful relations are

$$q^{\mu}(\bar{s}_L \Gamma b) = i\partial^{\mu}(\bar{s}_L \Gamma b) \tag{98}$$

which holds for $\bar{B}(p) \to \bar{K}^{(*)}(k)$ matrix elements, and

$$\partial_{u}(\bar{s}_{L}\Gamma b) = \bar{s}_{L}\Gamma \overleftarrow{D}_{u}b + \bar{s}_{L}\Gamma D_{u}b \tag{99}$$

or similar identities. Exactly three structures can be formed:

$$\bar{s}_L \gamma^\mu b$$
, $q_\nu \bar{s}_L \sigma^{\mu\nu} b$, $q^\mu \bar{s}_L b$ (100)

This can be seen as follows. For $\Gamma = 1$ in (97) the only possibility is the third operator in (100). For $\Gamma = \gamma^{\alpha}$ we obtain the first operator in (100) and

$$q^{\mu}q_{\nu}\bar{s}_{L}\gamma^{\nu}b = q^{\mu}(\bar{s}_{L}i \not\!\!\!\!D b + \bar{s}_{L}i \not\!\!\!D b) = m_{b}q^{\mu}\bar{s}_{L}b \tag{101}$$

which is equivalent to the third term in (100). For $\Gamma = \sigma^{\alpha\beta}$ we can write the second operator in (100). In addition, we could form

$$\varepsilon^{\mu\alpha\lambda\nu}q_{\alpha}\bar{s}_{L}\sigma_{\lambda\nu}b \sim q_{\alpha}\bar{s}_{L}\sigma^{\mu\alpha}\gamma_{5}b = q_{\alpha}\bar{s}_{L}\sigma^{\mu\alpha}b \tag{102}$$

which leads back to the same structure. This exhausts the possibilities and proves the completeness of the basis in (100) at the dimension-3 level.

We next consider operators of dimension 4. These have, in addition to the fields in (97), one covariant derivative acting on the strange quark. Their general form is

$$(\bar{s}_L \overleftarrow{D} \Gamma b)^{\mu} \tag{103}$$

Because of

$$\bar{s}_L \overleftarrow{D} \Gamma D_{\lambda} b = \partial_{\lambda} (\bar{s}_L \overleftarrow{D} \Gamma b) - \bar{s}_L \overleftarrow{D} \overleftarrow{D}_{\lambda} \Gamma b = -iq_{\lambda} (\bar{s}_L \overleftarrow{D} \Gamma b) + \dim 5$$
 (104)

terms with extra covariant derivatives acting on b do not lead to independent dimension-4 operators and may be disregarded.

For $\Gamma = 1$ in (103) we can write the operator $\bar{s}_L \overleftarrow{D}^{\mu} b$. Using the identity in (16) this operator can be expressed in terms of the basis in (100):

$$\bar{s}_L \overleftarrow{D}^{\mu} b = \frac{i}{2} m_b \bar{s}_L \gamma^{\mu} b - \frac{1}{2} q_{\nu} \bar{s}_L \sigma^{\mu\nu} b - \frac{i}{2} q^{\mu} \bar{s}_L b \tag{105}$$

The other possible structure can be reduced as

$$q^{\mu}q_{\nu}\bar{s}_{L}\overleftarrow{D}^{\nu}b = \frac{i}{2}(m_{b}^{2} - q^{2})q^{\mu}\bar{s}_{L}b$$
 (106)

For $\Gamma = \gamma^{\alpha}$ the dimension-4 operator has the form

$$\bar{s}_L \overleftarrow{D}^{\nu} \gamma^{\alpha} b \tag{107}$$

We list the possible contractions of (107) into a 4-vector together with the reduction to

the basis in (100), neglecting operators of dimension 5.

$$q^{\mu}q_{\nu}q_{\alpha}\bar{s}_{L}\overleftarrow{D}^{\nu}\gamma^{\alpha}b = \frac{i}{2}m_{b}(m_{b}^{2} - q^{2})q^{\mu}\bar{s}_{L}b$$

$$\tag{109}$$

$$-iq_{\nu}\bar{s}_{L}\overleftarrow{D}^{\nu}\gamma^{\mu}b = \bar{s}_{L}\overleftarrow{D}^{\nu}\gamma^{\mu}D_{\nu}b$$

$$= \frac{1}{2} \partial^{\nu} \partial_{\nu} (\bar{s}_{L} \gamma^{\mu} b) - \frac{1}{2} \bar{s}_{L} \gamma^{\mu} D^{\nu} D_{\nu} b = \frac{m_{b}^{2} - q^{2}}{2} \bar{s}_{L} \gamma^{\mu} b$$
 (110)

$$q_{\alpha}\bar{s}_{L}\overleftarrow{D}^{\mu}\gamma^{\alpha}b = \bar{s}_{L}\overleftarrow{D}^{\mu}i \not\!\!D b = m_{b}\bar{s}_{L}\overleftarrow{D}^{\mu}b \tag{111}$$

$$i\varepsilon^{\mu\alpha\lambda\nu}q_{\alpha}\bar{s}_{L}\overleftarrow{D}_{\lambda}\gamma_{\nu}b = q_{\alpha}\bar{s}_{L}\overleftarrow{D}_{\lambda}(\gamma^{\lambda}\gamma^{\mu}\gamma^{\alpha} - g^{\lambda\mu}\gamma^{\alpha} - g^{\alpha\mu}\gamma^{\lambda} + g^{\alpha\lambda}\gamma^{\mu})b =$$

$$= -q_{\alpha}\bar{s}_{L}\overleftarrow{D}^{\mu}\gamma^{\alpha}b + q^{\lambda}\bar{s}_{L}\overleftarrow{D}_{\lambda}\gamma^{\mu}b$$
(112)

The r.h.s. of (111) is reduced to the basic operators via (105). The terms on the r.h.s. of (112) lead back to the expressions (111) and (110).

For $\Gamma = \sigma^{\alpha\beta}$ the dimension-4 operator becomes

$$\bar{s}_L \overleftarrow{D}^{\lambda} \sigma^{\alpha\beta} b \tag{113}$$

We list again the possible contractions and their reduction to the operators in (100).

$$\bar{s}_L \overleftarrow{D}_{\lambda} \sigma^{\lambda \mu} b = i \bar{s}_L \overleftarrow{D}_{\lambda} (\gamma^{\lambda} \gamma^{\mu} - g^{\lambda \mu}) b = -i \bar{s}_L \overleftarrow{D}^{\mu} b \tag{114}$$

$$q_{\lambda}q_{\beta}\bar{s}_{L}\overleftarrow{D}^{\lambda}\sigma^{\mu\beta}b = im_{b}q_{\lambda}\bar{s}_{L}\overleftarrow{D}^{\lambda}\gamma^{\mu}b - iq_{\lambda}q^{\mu}\bar{s}_{L}\overleftarrow{D}^{\lambda}b$$
(115)

$$\varepsilon^{\mu\lambda\alpha\beta}\bar{s}_L \overleftarrow{D}_{\lambda}\sigma_{\alpha\beta}b \sim \bar{s}_L \overleftarrow{D}_{\lambda}\sigma^{\lambda\mu}b \tag{116}$$

The r.h.s. of (114) is given by (105). The r.h.s. of (115) is equivalent to (110) and (106), and (116) reduces to (114).

This completes the proof that all operators of dimension 3 and 4, (97) and (103), can be expressed in terms of (100) for $m_s = 0$. Taking current conservation into account leaves us with $\mathcal{O}_{3,1}^{\mu}$ and $\mathcal{O}_{3,2}^{\mu}$ in (96). Similar arguments hold if \bar{s}_L is replaced by \bar{s}_R . Since right-handed strange quarks come with a factor m_s , the two further operators $\mathcal{O}_{4,1}^{\mu}$ and $\mathcal{O}_{4,2}^{\mu}$ in (17) and (18) are obtained for $m_s \neq 0$.

Similarly to the derivations above it can be shown that dimension-5 operators of the form $\bar{s}_L D_\alpha D_\beta \Gamma b$, contracted to a Lorentz vector with index μ , can always be reduced to linear combinations of (100) and operators containing a factor of the gluon field strength $G_{\alpha\beta}$. The former terms correspond to dimension-3 operators with a purely kinematic power suppression. Genuine operators of dimension 5 therefore have the form $(\bar{s}_L G \Gamma b)^\mu$, as quoted in (19).

| m_K | f_K | $f_{ }[49]$ | $f_{\perp} [50]$ |
|-------------|-------------|--------------------------------|------------------|
| 0.496 | 0.16 | 0.218 | 0.185 |
| m_B | $m_{B_s^*}$ | f_B | λ_B |
| 5.28 | 5.41 | 0.2 | 0.350 |
| \bar{m}_b | \bar{m}_c | $\Lambda_{\overline{ m MS},5}$ | m_{K^*} |
| 4.2 | 1.3 | 0.225 | 0.894 |

Table 1: Input parameters in GeV. $f_{\perp} = f_{\perp}(1 \,\text{GeV})$.

B Numerical input

In this appendix we collect input we have used in numerical calculations. The numbers quoted are our central values. The form factors have an uncertainty of roughly $\pm 15\%$.

The $B \to K^*$ form factors are parametrized as [48]

$$A_0(q^2) = \frac{1.364}{1 - q^2/(5.28 \,\text{GeV})^2} - \frac{0.990}{1 - q^2/(36.78 \,\text{GeV}^2)}$$
(117)

$$V(q^2) = \frac{0.923}{1 - q^2/(5.32 \,\text{GeV})^2} - \frac{0.511}{1 - q^2/(49.40 \,\text{GeV}^2)}$$
(118)

The $B \to K$ form factor is parametrized as [22]

$$f_{+}(s) = f_{+}(0) \frac{1 - (b_0 + b_1 - a_0 b_0)s}{(1 - b_0 s)(1 - b_1 s)}, \qquad s \equiv \frac{q^2}{m_B^2}, \qquad b_0 \equiv \frac{m_B^2}{m_{B_s^*}^2}$$
 (119)

with the default choice $f_{+}(0) = 0.304$, $a_0 = 1.6$ and $b_1 = b_0$.

Further parameters are summarized in Table 1.

Acknowledgements

We thank the Galileo Galileo Institute for Theoretical Physics for hospitality and the INFN for partial support during the initial phase of this project. G.B. thanks the CERN Theory Division for support and hospitality while this paper was being completed. We thank Alexander Khodjamirian and Thomas Mannel for discussions and critical remarks. This work was supported in part by the DFG cluster of excellence 'Origin and Structure of the Universe' and by the DFG Graduiertenkolleg GK 1054.

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